13. Ivanov, M. Ia., On a variational problem of supersonic gas dynamics. Izv. Akad. Nauk SSSR, Mekh. Zhid. Gaza, №1, 1968.
14. Borisov, V. M., On a system of bodies with minimum wave drag. Inzh. Zh., Vol. 5, №6, 1965.
15. Shipilin, A. V., Optimal shapes of bodies with attached shock waves. Izv. Akad. Nauk SSSR, Mekh. Zhid. Gaza, №4, 1966.
16. Katskova, O. N., Calculation of equilibrium gas flows in supersonic nozzles. Tr. Vychisl. Tsentr. Akad. Nauk SSSR, M., 1964.
17. Kraiko, A. N. and Osipov, A. A., On the solution of variational problems of supersonic flows of gas with foreign particles. PMM Vol. 32, №4, 1968.
18. Kraiko, A. N. and Osipov, A. A., On the determination of the shape of a supersonic nozzle taking into consideration the variation of aircraft flight conditions. PMM Vol. 34, №6, 1970.
19. Chernyi, G. G., Introduction to hypersonic flow. Academic Press, New York and London, 1961.
20. Osipov, A. A., On the solution of variational problems of the gas dynamics of supersonic nonequilibrium flow. Izv. Akad. Nauk SSSR, Mekh. Zhid. Gaza, №1, 1969.

Translated by M. D. V. D.

ON ONE TYPE OF INTERACTION OF THE BOUNDARY LAYER AND THE OUTER (INVISCID) STREAM AT SUPERSONIC SPEEDS

PMM Vol. 35, No. 4, 1971, pp. 633-637

B. M. BULAKH
(Leningrad)

(Received January 15, 1971)

The problem is considered of unsymmetric steady flow past a circular cone in a uniform supersonic stream of viscous gas at high Reynolds number R. It was shown in [1] that in many cases the solution of the problem of inviscid flow past a cone is such that normal derivatives of the density (and temperature) and of the velocity components of the gas tangent to the surface become infinite at the surface of the cone. In these cases, it follows from the condition of matching the solution for inviscid flow past the cone (which is regarded as the first term of an asymptotic expansion of the solution of the complete problem in powers of $\varepsilon = R^{-1/2}$ outside the boundary layer) with the solution of the problem in the boundary layer that supplementary terms appear in the latter solution, which may give a significant correction to the results of the usual boundary-layer theory. It is shown (in the case of a laminar boundary layer) that these supplementary terms are self-similar; and a strict formulation is given of the problem for their determination.

1. We consider steady flow past a circular cone of semi-vertex angle β in a uniform supersonic stream of viscous gas at angle of attack α. In a system of coordinates in

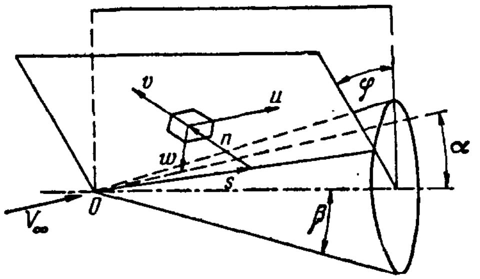

Fig. 1

which s is measured from the vertex of the cone along generators, n is normal to the body, and the angle φ determines the meridional plane (see Fig. 1), the equations of continuity, momentum (Navier-Stokes equations) and energy, and the equation of state of the gas have the form

$$\left(\frac{\rho u}{h}\right)_s + \left(\frac{\rho v}{h}\right)_n + (\rho w)_\varphi = 0 \tag{1.1}$$

$$\begin{aligned}\varepsilon^{-2}\{\rho\,[uu_s + vu_n + hw\,(u_\varphi - \sin\beta w)] + p_s\} = \{(\lambda + 2\mu)\,u_s + \\ + \lambda\,[v_n + h\,(w_\varphi + u\sin\beta + v\cos\beta)]\}_s + [\mu\,(v_s + u_n)]_n + \\ + h\,\{\mu\,[h\,(u_\varphi - w\sin\beta) + w_s]\}_\varphi + \mu h\,\{2\sin\beta\,[u_s - \\ - h\,(w_\varphi + u\sin\beta + v\cos\beta)] + \cos\beta\,(v_s + u_n)\}\end{aligned} \tag{1.2}$$

$$\begin{aligned}\varepsilon^{-2}\{\rho\,[uw_s + vw_n + hw\,(w_\varphi + u\sin\beta + v\cos\beta)] + hp_\varphi\} = \{\mu\,[h\,(u_\varphi - w\sin\beta) + \\ + w_s]\}_s + \{\mu\,[w_n + h\,(v_\varphi - w\cos\beta)]\}_n + h\,[(\lambda + 2\mu)\,h\,(w_\varphi + u\sin\beta + v\cos\beta) + \\ + \lambda\,(v_n + u_s)]_\varphi + 2h\mu\,\{\sin\beta\,[h(u_\varphi - w\sin\beta) + w_s] + \cos\beta\,[h(v_\varphi - w\cos\beta) + w_n]\}\end{aligned} \tag{1.3}$$

$$\begin{aligned}\varepsilon^{-2}\{\rho\,[uv_s + vv_n + hw\,(v_\varphi - \cos\beta w)] + p_n\} = [\mu\,(v_s + u_n)]_s + \\ + \{(\lambda + 2\mu)\,v_n + \lambda\,[u_s + h\,(w_s + u\sin\beta + v\cos\beta)]\}_n + \\ + h\,\{\mu\,[w_n + h\,(v_\varphi - w\cos\beta)]\}_\varphi + \mu h\,\{\sin\beta\,(v_s + u_n) + \\ + 2\cos\beta\,[v_n - h\,(w_\varphi + u\sin\beta + v\cos\beta)]\}\end{aligned} \tag{1.4}$$

$$\begin{aligned}\varepsilon^{-2}\,[\rho\,(uT_s + vT_n + whT_\varphi) - (up_s + vp_n + whp_\varphi)] = \\ = \sigma^{-1}\,[(\mu T_s)_s + (\mu T_n)_n + h^2\,(\mu T_\varphi)_\varphi] + \Phi\end{aligned} \tag{1.5}$$

$$\begin{aligned}\Phi = \mu\,\{2\,[u_s^2 + v_n^2 + h^2\,(w_\varphi + u\sin\beta + v\cos\beta)^2] + \\ + (v_s + u_n)^2 + [w_n + h\,(v_\varphi - w\cos\beta)]^2 + [w_\varphi + h\,(u_\varphi - w\sin\beta)]^2 + \\ + \lambda\,[u_s + v_n + h\,(w_\varphi + u\sin\beta + v\cos\beta)]^2\end{aligned}$$

$$p = \frac{\gamma - 1}{\gamma}\rho T, \quad \mu = \mu\,(T), \quad \lambda = \lambda\,(T)$$

$$\varepsilon = \frac{1}{\sqrt{R}}, \quad R = \frac{V_0 l_0 \rho_0}{\mu_0}, \quad h = (s\sin\beta + n\cos\beta)^{-1} \tag{1.6}$$

Here u, v, w are the components of the velocity vector in the direction of increasing s, n, φ respectively, ρ is the density, p the pressure, T the temperature, σ the Prandtl number, μ and λ the coefficients of viscosity, γ the adiabatic index, and R the Reynolds number formed from the characteristic parameters of the flow. Derivatives are indicated by subscripts; for example, $u_n = \partial u/\partial n$. In Eqs. (1.1)–(1.6) all lengths are referred to l_0, velocities to V_0, density to ρ_0, pressure to $\rho_0 V_0^2$, temperature to $T_0 = V_0^2 c_p^{-1}$ (where c_p is the specific heat at constant pressure), and the coefficients of viscosity to μ_0. The notation for dimensionless quantities is the same as for dimensional ones.

2. To solve the problem of flow past a body in the case $\varepsilon \ll 1$ we distinguish the boundary layer – a region of thickness $O(\varepsilon)$ immediately adjacent to the surface of the body – and the "outer flow" region. In the latter region the solution of the system of equations (1.1) – (1.6) is, in the simplest case, sought in the form of an asymptotic expansion

$$f(s, n, \varphi, \varepsilon) \sim F_1(s, n, \varphi) + \varepsilon F_2(s, n, \varphi) + \ldots \tag{2.1}$$

Here, f stands for u, v, w, p, ρ or T. In the region of the boundary layer, where viscous forces are of the same order as inertia forces, we introduce the variable $N = n\varepsilon^{-1}$, and the solution is sought in the form of the asymptotic expansion

$$f(s, n, \varphi, \varepsilon) \sim f_1(s, N, \varphi) + \varepsilon f_2(s, N, \varphi) + \ldots \tag{2.2}$$
$$v \sim \varepsilon v_1 + \varepsilon^2 v_2$$

where f stands for u, w, p, ρ or T. The equations for the first terms of the expansion (2.1) are the Euler equations, and those for the first terms of the expansion (2.2) are Prandtl's boundary-layer equations. The complete solution of the problem is obtained by matching the expansions (2.1) and (2.2) in a certain overlap region [2].

3. For the case under consideration of a circular cone, the terms $F_1(s, n, \varphi)$ in the expansion (2.1), corresponding to the solution of the problem of inviscid flow past the cone, have (for conical flow) the following form near the surface of the cone ($n = 0$) [1]

$$\rho = \rho^* + A(n/s)^B + \ldots, \qquad p = p^* + \rho^* w^{*2} \operatorname{ctg}\beta\,(n/s) + O[(n/s)^{B+1}]$$
$$u = u^* + D\left(\frac{n}{s}\right)^B + \ldots, \qquad v = -\left[u^* + \frac{1}{\rho^*\sin\beta}(\rho^* w^*)_\varphi\right]\left(\frac{n}{s}\right) + O\left[\left(\frac{n}{s}\right)^{B+1}\right]$$
$$w = w^* + \frac{1}{w^*}\left(\frac{\gamma}{\gamma-1}\frac{p^*}{\rho^{*2}}A - u^* D\right)\left(\frac{n}{s}\right)^B + \ldots \tag{3.1}$$

Here $f^* = f^*(\varphi) = f(n/s, \varphi)/_{n=0}$ (the dots indicate terms of higher order of smallness in n/s than those shown), A and D are certain functions of φ, and the constant B is determined by the equation

$$B = \frac{2\tau}{2+\tau}, \qquad \tau = \frac{u^*_{\varphi\varphi}(\pi)}{u^*(\pi)\sin^2\beta} \tag{3.2}$$

and in most cases $B < 1$. Henceforth we consider the case $B < 1$.

4. The expansions (2.1) and (2.2) are matched in the "overlap region" by writing (2.1) for small n, and then replacing n by $n = N\varepsilon$; the result should be expression (2.2) written for large N. Setting $n = N\varepsilon$ in Eq. (3.1) gives

$$\rho = \rho^* + \varepsilon^B A(N/s)^B + \ldots, \qquad p = p^* + O(\varepsilon)$$
$$u = u^* + \varepsilon^B D(N/s)^B + \ldots, \qquad v = O(\varepsilon) \tag{4.1}$$
$$w = w^* + \varepsilon^B \frac{1}{w^*}\left(\frac{\gamma}{\gamma-1}\frac{p^*}{\rho^{*2}}A - u^* D\right)(N/s)^B + \ldots$$

The first terms in (4.1), written explicitly, yield boundary conditions at large N for the terms $f_1(s, N, \varphi)$ in the expansions (2.2), which determine the usual solution in the boundary layer. (The matching condition for v is satisfied automatically). The secondary terms in (4.1), of order ε^B, show that in the case $B < 1$ the expansion (2.2) must be replaced by the expansion

$$f(s, n, \varphi, \varepsilon) \sim f_1(s, N, \varphi) + \varepsilon^B f_2(s, N, \varphi) + \ldots \qquad (4.2)$$

It can be shown that the terms $f_2(s, N, \varphi)$ satisfy linear equations which are obtained by perturbing the usual Prandtl boundary-layer equations.

For $B \ll 1$ the quantity ε^B will not be small (for example, for $M_\infty = 7$, $\beta = 30°$, $\alpha = 5°$, $B \approx 0.075$, and for $R = 10^6$, $\varepsilon^B \approx 0.6$) ; therefore terms of $O(\varepsilon^B)$ in (4.2) may give a significant correction to the results of conventional boundary-layer theory. The interaction between the inviscid flow and the boundary layer described above is similar to the vortical interaction at hypersonic speeds, but there the interaction is caused by strong curvature of the bow shock wave, whereas here no simple physical cause is evident for the origin of the interaction.

5. Considering what has been said above, and also the fact that the solution of the usual boundary-layer problem for the circular cone is self-similar [3], we will seek the solution of Eqs. (1.1) – (1.6) in the boundary-layer region (outside some neighborhood of the vertex of the cone) in the form

$$u = u_1(\zeta, \varphi) + \varepsilon^B s^{-1/_2 B} u_2^\circ(\zeta, \varphi) + o(\varepsilon^B)$$
$$w = w_1(\zeta, \varphi) + \varepsilon^B s^{-1/_2 B} w_2^\circ(\zeta, \varphi) + o(\varepsilon^B)$$
$$v = \varepsilon s^{-1/_2}[v_1^\circ(\zeta, \varphi) + \varepsilon^B s^{-1/_2 B} v_2^\circ(\zeta, \varphi) + o(\varepsilon^B)], \qquad p = p_1(\varphi) + o(\varepsilon^B)$$
$$\rho = \rho_1(\zeta, \varphi) + \varepsilon^B s^{-1/_2 B} \rho_2^\circ(\zeta, \varphi) + o(\varepsilon^B)$$
$$[T = t_1(\zeta, \varphi) + \varepsilon^B s^{-1/_2 B} t_2^\circ(\zeta, \varphi) + o(\varepsilon^B)], \quad \zeta = N s^{-1/_2}, \quad N = n\varepsilon^{-1} \qquad (5.1)$$

Substituting the expansions (5.1) into the system of equations (1.1) – (1.6) and equating terms of the same order in ε we obtain the following systems of equations for the functions with subscripts 1 and 2:

$$(\rho_1 w_1)_\varphi + \sin\beta\,[\rho_1 u_1 - {}^1/_2\zeta(\rho_1 u_1)_\zeta + (\rho_1 v_1^\circ)_\zeta] = 0 \qquad (5.2)$$

$$\rho_1\left[w_{1\zeta}(v_1^\circ - {}^1/_2\zeta u_1) + \frac{w_1}{\sin\beta}(w_{1\varphi} + u_1\sin\beta)\right] + \frac{p_{1\varphi}}{\sin\beta} = (\mu_1 w_{1\zeta})_\zeta$$

$$\rho_1\left[u_{1\zeta}(v_1^\circ - {}^1/_2\zeta u_1) + \frac{w_1}{\sin\beta}u_{1\varphi} - w_1^2\right] = (\mu_1 u_{1\zeta})_\zeta$$

$$\rho_1\left[t_{1\zeta}(v_1^\circ - {}^1/_2\zeta u_1) + \frac{w_1}{\sin\beta}t_{1\varphi}\right] - \frac{w_1}{\sin\beta}p_{1\varphi} = \sigma^{-1}(\mu_1 t_{1\zeta})_\zeta + \mu_1(u_{1\zeta}^2 + w_{1\zeta}^2)$$

$$p_1 = \frac{\gamma - 1}{\gamma}\rho_1 t_1, \qquad \mu_1 = \mu(t_1), \qquad p_{1\zeta} = 0$$

$$\sin\beta\,[(1 - {}^1/_2 B)(\rho_1 u_2^\circ + u_1\rho_2^\circ) - {}^1/_2\zeta(\rho_1 u_2^\circ + u_1\rho_2^\circ)_\zeta +$$
$$+ (\rho_1 v_2^\circ + \rho_2^\circ v_1^\circ)_\zeta] + (\rho_1 w_2^\circ + \rho_2^\circ w_1)_\varphi = 0 \qquad (5.3)$$

$$\rho_2^\circ\left[w_{1\zeta}(v_1^\circ - {}^1/_2\zeta u_1) + \frac{w_1}{\sin\beta}(w_{1\varphi} + u_1\sin\beta)\right] + \rho_1\Big\{-{}^1/_2\,[B u_1 w_2^\circ +$$

$$+ \zeta(u_1 w_{2\zeta}^\circ + u_2^\circ w_{1\zeta})] + v_1^\circ w_{2\zeta}^\circ + v_2^\circ w_{1\zeta} + \frac{w_2^\circ}{\sin\beta}(w_{1\varphi} + u_{1\varphi}\sin\beta) +$$

$$+\frac{w_1}{\sin\beta}(w_{2\varphi}{}^\circ + u_2{}^\circ \sin\beta)\Big\} = (\mu_1 w_{2\zeta}{}^\circ)_\zeta + (\mu_t w_{1\zeta} t_2{}^\circ)_\zeta, \quad \left[\mu_t = \left(\frac{d\mu}{dT}\right)_{T=t_1}\right]$$

$$\rho_2{}^\circ\left[u_{1\zeta}(v_1{}^\circ - {}^1/_2\zeta u_1) + \frac{w_1}{\sin\beta}u_{1\varphi} - w_1{}^2\right] + \rho_1\Big\{-{}^1/_2\,[Bu_1u_2{}^\circ +$$

$$+\zeta(u_1u_{2\zeta}{}^\circ + u_{1\zeta}u_2{}^\circ)] + v_1{}^\circ u_{2\zeta}{}^\circ + v_2{}^\circ u_{1\zeta} + \frac{1}{\sin\beta}(w_1u_{2\varphi}{}^\circ + w_2{}^\circ u_{1\varphi}) -$$

$$-2w_1u_2{}^\circ\Big\} = (\mu_1 u_{2\zeta}{}^\circ)_\zeta + (\mu_t t_2{}^\circ u_{1\zeta})_\zeta$$

$$\rho_2{}^\circ\left[t_{1\zeta}(v_1{}^\circ - {}^1/_2\zeta u_1) + \frac{w_1}{\sin\beta}t_{1\varphi}\right] + \rho_1\Big\{-{}^1/_2\,[Bu_1t_2{}^\circ +$$

$$+\zeta(u_1t_{2\zeta}{}^\circ + u_2{}^\circ t_{1\zeta})] + v_1{}^\circ t_{2\zeta}{}^\circ + v_2{}^\circ t_{1\zeta} + \frac{1}{\sin\beta}(w_1t_{2\varphi}{}^\circ + w_2{}^\circ t_{1\varphi})\Big\} -$$

$$-\frac{1}{\sin\beta}p_{1\varphi}w_2{}^\circ = \sigma^{-1}[(\mu_1 t_{2\zeta}{}^\circ)_\zeta + (\mu_t t_2{}^\circ t_{1\zeta})_\zeta] +$$

$$+2\mu_1(u_{1\zeta}u_{2\zeta}{}^\circ + w_{1\zeta}w_{2\zeta}{}^\circ) + \mu_t t_2{}^\circ(u_{1\zeta}{}^2 + w_{1\zeta}{}^2)$$

$$\rho_2{}^\circ = -\frac{\rho_1}{t_1}t_2{}^\circ$$

The boundary conditions for the systems of equations (5.2) and (5.3) follow from the conditions for flow past the cone and Eqs. (4.1), and have the form

$$\zeta = 0, \quad u_1 = v_1{}^\circ = w_1 = 0, \quad t_1 = T_w \qquad (\text{or} \quad t_{1\zeta} = 0) \tag{5.4}$$

(where T_w is the temperature of the cone surface)

$$\zeta \to +\infty, \quad u_1 \to u^*(\varphi), \quad w_1 \to w^*(\varphi), \quad \rho_1 \to \rho^*(\varphi)$$

$$\left[t_1 \to \frac{p_1{}^*(\varphi)}{\rho_1{}^*(\varphi)}\frac{\gamma}{\gamma-1}, \quad v_1{}^\circ \sim O(\zeta)\right]$$

$$\zeta = 0, \quad u_2{}^\circ = v_2{}^\circ = w_2{}^\circ = 0, \quad t_2{}^\circ = 0 \qquad (\text{or} \quad t_{2\zeta}{}^\circ = 0) \tag{5.5}$$

$$\zeta \to +\infty, \qquad u_2{}^\circ \sim D(\varphi)\,\zeta^B, \quad \rho_2{}^\circ \sim A(\varphi)\,\zeta^B$$

$$w_2{}^\circ \sim \frac{1}{w^*(\varphi)}\left[\frac{\gamma}{\gamma-1}\frac{p^*(\varphi)}{(\rho^*(\varphi))^2}A(\varphi) - u^*(\varphi)D(\varphi)\right]\zeta^B$$

$$\left[t_2{}^\circ \sim -\frac{t_1}{\rho_1}A\zeta^B, \quad v_2{}^\circ \sim O(\zeta^{B+1})\right]$$

6. The system of equations (5.2) with boundary conditions (5.4) determines the known solution of Prandtl's equations for the cone. Numerical integration of a system of equations equivalent to the system (5.2) was carried out in [3]. We note that the system of equations (5.2) was considered as evolutionary in [3], with the angle φ playing the role of time. The flow variables are found first in the plane $\varphi = \pi$, and then for $0 < \varphi \leqslant \pi$. The system of equations (5.3) with boundary conditions (5.5) can also be considered as evolutionary, where for $\varphi = \pi$

$$u_2{}^\circ = w_2{}^\circ = v_2{}^\circ = \rho_2{}^\circ = t_2{}^\circ = 0, \quad u_{2\varphi}{}^\circ = w_{2\varphi}{}^\circ = v_{2\varphi}{}^\circ = \rho_{2\varphi}{}^\circ = t_{2\varphi}{}^\circ = 0$$

BIBLIOGRAPHY

1. Bulakh, B. M., On the vortical layer on a circular cone. Izv. Akad. Nauk SSSR, Mekh. Zhid. Gaza, №1, 1971.
2. Van Dyke, M., Perturbation Methods in Fluid Mechanics. Akademic Press, N.Y., 1964.
3. Vvedenskaia, N. D., Calculation of the boundary layer arising for flow past a cone at angle of attack. J. U.S.S.R. Comput. Math. and Math. Phys., Pergamon Press, Vol. 6, №2, 1966.

Translated by M. D. V. D.

THE ONSET OF AUTO-OSCILLATIONS IN A FLUID

PMM Vol. 35, No. 4, 1971, pp. 638-655
V. I. IUDOVICH
(Rostov-on-Don)
(Received July 22, 1970)

The onset of auto-oscillations at transition of the Reynolds number (or any other parameter defining the steady motion of a viscous incompressible fluid) through its critical value is investigated.

Landau in [1] (see, also, [2, 3]) considered the onset of the periodic auto-oscillation mode to be the first stage of transition from a laminar to a turbulent flow of a fluid. His method, developed also by Meksyn, Stuart and Watson (see [4–7]), implies the knowledge of the eigenvectors of the linearized (with respect to the basic laminar mode at a given Reynolds number) Navier-Stokes operator to which (according to the linear theory) correspond increasing perturbations. A system of ordinary nonlinear differential equations is derived for the determination of the Fourier coefficients of the velocity field. The calculation of the right-hand sides of equations of this system is, however, somewhat involved. Owing to this, this method had not, so far, provided final results in specific cases, such as, for example, the Poiseuille flow in a channel. The Landau method is clearly more suitable for investigating the onset of a periodic mode rather than for the calculation of a stabilized one.

Here the onset of auto-oscillations is analyzed by the Liapunov-Schmidt method described in [8, 9]. The branching out of periodic solutions of systems of ordinary differential equations is considered in [10], where references to earlier works are cited. The generation of a cycle is considered in [10, 11] for a system of ordinary differential equations, while [12–14] deal with the special case of Galerkin equations approximating the Navier-Stokes system. Certain statements related to the complete Navier-Stokes equations are also formulated in [13, 14].

A comprehensive statement of the problem and basic definitions are given in Sect. 1; an a priori estimate of possible auto-oscillation modes is presented (Lemma 1.2), and it is shown that only the critical value of a parameter can be a point of branching out of the system (Lemma 1.3).

This is followed by the analysis of supplementary conditions for the actual